

# Three-monotone interpolation\*

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Rev. 10/IV/13 JM

## Abstract

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is called  $k$ -monotone if it is  $(k - 2)$ -times differentiable and its  $(k - 2)$ nd derivative is convex. A point set  $P \subset \mathbb{R}^2$  is  $k$ -monotone interpolable if it lies on a graph of a  $k$ -monotone function. These notions have been studied in analysis, approximation theory etc. since the 1940s.

We show that 3-monotone interpolability is very non-local: we exhibit an arbitrarily large finite  $P$  for which every proper subset is 3-monotone interpolable but  $P$  itself is not. On the other hand, we prove a Ramsey-type result: for every  $n$  there exists  $N$  such that every  $N$ -point  $P$  with distinct  $x$ -coordinates contains an  $n$ -point  $Q$  such that  $Q$  or its vertical mirror reflection are 3-monotone interpolable. The analogs for  $k$ -monotone interpolability with  $k = 1$  and  $k = 2$  are classical theorems of Erdős and Szekeres, while the cases with  $k \geq 4$  remain open.

We also investigate the computational complexity of deciding 3-monotone interpolability of a given point set. Using a known characterization, this decision problem can be stated as an instance of polynomial optimization and reformulated as a semidefinite program. We exhibit an example for which this semidefinite program has only doubly exponentially large feasible solutions, and thus known algorithms cannot solve it in polynomial time. While such phenomena have been well known for semidefinite programming in general, ours seems to be the first such example in polynomial optimization, and it involves only univariate quadratic polynomials.

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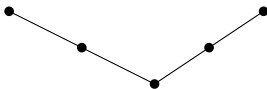
\*This research was started at the 3rd KAMÁK workshop held in Vranov nad Dyjí, Czech Republic, September 15-20, 2013, which was supported by the grant SVV-2013-267313 (Discrete Models and Algorithms). J.C. was also supported by this grant. J.M. was supported by the ERC Advanced Grant No. 267165. P.P. was supported by the grant SVV-2014-260107

# 1 Introduction

**Generalizing two theorems of Erdős and Szekeres.** This research was inspired by two famous 1935 theorems of Erdős and Szekeres [ES35]. The first one asserts that for every  $n$  there is  $N$  such that every sequence  $P$  of  $N$  points in the plane with increasing  $x$ -coordinates contains an  $n$ -point nonincreasing or nondecreasing subsequence (see, e.g., Steele [Ste95] for six nice proofs and some applications), and the second theorem makes an analogous statement about the existence of an  $n$ -point convex or concave subsequence (see, e.g., Morris and Soltan [MS00] for proofs and a survey of developments around this result).

For our purposes, a *nondecreasing sequence* can be defined as one lying on the graph of a nondecreasing function  $\mathbb{R} \rightarrow \mathbb{R}$ , and similarly for nonincreasing, convex, and concave sequences. Eliáš and Matoušek [EM13] suggested a generalization where one looks for a subsequence lying on the graph of a function whose  $k$ th derivative is nonnegative or nonpositive. Here we consider this question but in a slightly different and technically more convenient formulation. (Let us remark that a number of other generalizations of the Erdős–Szekeres theorems have recently been considered [FPSS12, CFP<sup>+</sup>13, BM14, Suk13, EMRS14].)

**$k$ -monotone functions.** The following five-point set



lies on the graph of a convex function but not on the graph of a convex twice differentiable function. This illustrates that the requirement as above, with a function whose  $k$ th derivative is nonnegative or nonpositive, is not technically quite suitable.

In [EM13] this kind of issues was circumvented by assuming sufficiently general position of  $P$ . However, there is a well-established notion of  $k$ -monotonicity of a function, which seems perfectly suitable for our purposes and does not require any general position assumption.

Namely, for  $k \geq 2$ , a function  $f$  is  *$k$ -monotone* on an open interval  $I$  if its  $(k - 2)$ nd derivative  $f^{(k-2)}$  (exists and) is convex on  $I$ . (With some fantasy, this definition can also be applied for  $k = 1$  and leads to the usual notion of a nondecreasing function.)

Note that  $k$ -monotonicity is of the “nondecreasing” kind, while the corresponding “nonincreasing” notion has  $f^{(k-2)}$  concave. The term “ $k$ -monotone” may thus be somewhat confusing in this respect, since “monotone function” usually means nondecreasing *or* nonincreasing, but it seems well established in the literature.

The notion of  $k$ -monotonicity goes back to Schoenberg’s 1941 abstract [Sch41], preceded by a still older notion of a *completely monotone function*. It has been studied from various angles in a number of papers in relation to integral representations of functions, approximation theory, probability, etc. We refer

to Williamson [Wil56] for an early study<sup>1</sup> and to Pečarić et al. [PPT92] and Roberts and Varberg [RV73] for various properties and applications; for our investigations we mostly rely on Kopotun and Shadrin [KS03].

**A Ramsey-type result for 3-monotone interpolability.** Let us call a set  $P \subset \mathbb{R}^2$  *k-monotone interpolable* if it lies on a graph of a *k-monotone* function. The question about generalizing the Erdős–Szekeres theorems to *k-monotonicity* can be stated as follows:

**Question 1.1.** *For which  $k \geq 3$  does the following hold? For every integer  $n$  there exists  $N = N_k(n)$  such that every  $N$ -point  $P \subset \mathbb{R}^2$  with distinct  $x$ -coordinates contains an  $n$ -point subset  $Q$  such that  $Q$  or  $Q^\uparrow$  is *k-monotone interpolable* (where  $Q^\uparrow$  denotes the mirror reflection of  $Q$  about the  $x$ -axis).*

In Section 3 we provide a positive answer for  $k = 3$ .

**Theorem 1.2.** *The statement in Question 1.1 holds for  $k = 3$ .*

Unfortunately, our proof does not seem to generalize to any larger  $k$ , and so Question 1.1 remains open for  $k \geq 4$ .

**A nonlocal behavior of 3-monotone interpolability.** An obvious necessary condition for a set  $Q$  to be *k-monotone interpolable* is that every  $(k + 1)$ -tuple in  $Q$  be *k-monotone interpolable*, and for  $k \leq 2$  it is easy to check that this is also sufficient.

In earlier versions of [EM13], it was conjectured that the condition should be sufficient for all  $k \geq 3$ . If this were the case, then Theorem 1.2 would follow immediately from Ramsey’s theorem for fourtuples.

However, Rote found a counterexample for  $k = 3$  (reproduced in [EM13]): a six-point set  $P$  for which all fourtuples are 3-monotone interpolable, but  $P$  itself is not. Later we learned that a similar example was known earlier [KS03, Example 5.3].

In Section 4.1 we provide a much stronger example showing that 3-monotone interpolability is a completely global property.

**Theorem 1.3.** *For every even  $n \geq 4$  there exists an  $n$ -point  $P \subset \mathbb{R}^2$  that is not 3-monotone interpolable, but for which every proper subset is 3-monotone interpolable.*

This, in our opinion, makes Theorem 1.2 somewhat surprising and Question 1.1 for  $k \geq 4$  interesting.

It is straightforward to extend our proof of Theorem 1.3 to yield an analogous result for every *odd*  $k \geq 3$ . The case of even  $k$  seems somewhat more problematic, although we believe that the difficulties should not be unsurmountable.

**The algorithmic question.** We also investigate the computational complexity of the question, Given a finite  $P$  in the plane, is it *k-monotone interpolable*?

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<sup>1</sup>Let us remark that some of the literature, especially older one such as [Sch41, Wil56], the definition of *k-monotonicity* is somewhat different, also involving requirements on lower-order derivatives, but the essence of the notion remains the same. The term *k-convex* is also used instead of *k-monotone*.

This is a numerical problem, and so it is important to specify the model of computation, and also to distinguish exact and approximate version of the question.

We will use the *bit model* (or *Turing machine model*) of computation, where one counts the number of bit operations; thus, for example, the addition of two  $b$ -bit numbers takes time proportional to  $b$ . We assume that the coordinates of the points of the input set  $P$  are rational numbers, and the size of  $P$  is measured as the number of bits in its binary encoding (each of the rational coordinates is encoded by the numerator and denominator written in binary). See, e.g., Grötschel, Lovász, and Schrijver [GLS88] for more details on this model of computation.

Let us remark that for geometric computations, the *real RAM*, or *Blum–Shub–Smale*, model is also used in many papers, where arithmetic operations with arbitrary real numbers are allowed at unit cost. However, for testing  $k$ -monotone interpolability, we believe that this model is inadequate, since as we will show, a natural algorithm for this testing needs to deal with numbers having exponentially many digits.

Kopotun and Shadrin [KS03] provided a characterization of  $k$ -monotone interpolability, which we will recall in Section 2 below. Using this characterization and methods of polynomial optimization, as discussed e.g. in Lasserre’s book [Las10], one can write down a semidefinite program that is feasible if and only if the given point set  $P$  is *not* 3-interpolable. (We will provide a brief discussion of semidefinite programming and basic references in Section 5.1.)

In our experience, many people in theoretical computer science regard semidefinite programs more or less automatically as polynomial-time solvable problems. (Some of the authors certainly did belong among these people before working on the present paper.) Indeed, many introductory texts and classes may make this impression, although they usually point out that the known polynomial-time algorithms solve semidefinite programs only approximately.

However, for the polynomiality claim to be true, one also needs to assume that, if the semidefinite program in question is feasible at all, it has a feasible solution with norm bounded by an integer  $R$  with polynomially many bits (polynomially in the size of the input). It is known that such a bound need not hold in general and that the smallest feasible solution may need exponentially many bits, but in many applications of semidefinite programming, e.g., in combinatorial optimization, it is obvious that such a pathology cannot occur.

In contrast, for the semidefinite program mentioned above corresponding to 3-monotone interpolability, we found that there are simple input point sets that do force the smallest feasible solution to have exponentially many digits. This result, Corollary 5.2 below, is based on the following example.

**Theorem 1.4.** *Let  $P_i = \{z, p_0, p_1, \dots, p_{2m+1}, q\}$ , where  $z = (-1, 0)$ ,  $p_j = (j, j^3)$  for  $j = 0, 1, \dots, 2m + 1$ , and  $q = (2m + 2, (2m + 2)^3 - 6)$ . Let  $P'_m = (P_m \setminus \{q\}) \cup \{q'\}$ , where  $q'$  is  $q$  shifted upwards by  $2 \cdot 2^{-2^m}$ . Then  $P'_m$  is 3-monotone interpolable, while  $P_m$  is not.*

The best known algorithm for deciding feasibility of an arbitrary semidefinite program we could find in the literature is due to Porkolab and Khachiyan

[PK97], and it has exponential complexity (more precisely, the time complexity is at most  $\exp(O(s \log s))$ , where  $s$  is the input size). This also yields the best complexity of an exact algorithm for testing  $k$ -monotone interpolability we are aware of (another algorithm of comparable complexity can be obtained from algorithms for deciding sentences in the first-order theory of the reals, which are discussed, e.g., in book Basu, Pollack, and Roy [BPR03], but here we will not consider this alternative approach).

**Future work.** We consider the Ramsey-theoretic question, about the existence of a large  $k$ -monotone interpolable subset in any sufficiently large point set, interesting and unusual in the context of geometric Ramsey theory, because of the nonlocal nature of  $k$ -monotone interpolability. The open case  $k \geq 4$  seems to need a new idea. Another question is estimating the order of magnitude of the Ramsey function  $N_3(n)$ .

On the computational side, the problem of (exact) testing  $k$ -monotone interpolability can be regarded as a simple concrete instance of polynomial optimization in the spirit of [Las10]. Thus, it would be very interesting to obtain stronger hardness results, or possibly an algorithm with provably subexponential complexity.

For semidefinite programming, there is a lower bound result of Tarasov and Vyalyi [TV08]: the problem of deciding feasibility of a semidefinite program (exactly) is at least as hard as the following problem: given an integer arithmetic circuit without inputs, determine the sign of its output. This is a problem of basic importance for many complexity questions of numerical mathematics (see, e.g., Allender et al. [ABKPM09]), and its complexity status is unknown and probably very challenging to determine. Can an analog of the Tarasov–Vyalyi result be obtained for some simple case of polynomial optimization, such as the non-positivity problem (stated later as Problem 5.1)? Or perhaps even for the very specific case of testing 3-monotone interpolability?

According to Ramana [Ram97], given a semidefinite program  $\Pi$ , one can construct another semidefinite program, the *Ramana dual* of  $\Pi$ , that is feasible iff  $\Pi$  is infeasible, and whose input size is bounded by a polynomial in the input size of  $\Pi$ . Thus, testing feasibility of a semidefinite program is, in this sense, symmetric with respect to the YES and NO answers; for example, it either belongs to both NP and co-NP, or it is outside of both NP and co-NP. Can a similar result be obtained for polynomial optimization, and/or for 3-monotone interpolability?

Our example in Theorem 1.4 indicates that at least the “obvious” certificates of 3-monotone *noninterpolability* are not of polynomial size. Is there a polynomial-size certificate for 3-monotone *interpolability*, or some result indicating that such a certificate is unlikely to be found?

One might also seek an “elementary” algorithm for deciding 3-monotone interpolability, say one trying to combine an interpolant from suitable parabolic arcs.

Finally, in spite of our negative examples, one may hope that the  $k$ -monotone interpolability problem, at least for not too many points, is “usually” solvable in practice by running a semidefinite solver on the semidefinite program set up

in Section 5.1. For this to have at least some theoretical foundation, it would be good to have an approximation result of the following kind: *There is an algorithm that, given  $k$ , a point set  $P$ , and a parameter  $\varepsilon > 0$ , returns YES or NO, and runs in time polynomial in  $k$ , the input size of  $P$ , and  $\log \frac{1}{\varepsilon}$ . If the answer is NO, then  $P$  is not  $k$ -monotone interpolable, and if the answer is YES, then there is a  $k$ -monotone interpolable set  $P'$  that can be obtained from  $P$  by shifting every point up or down by at most  $\varepsilon$ .*

Currently we do not have such a result. There are theoretical bounds, based on the ellipsoid method, on the complexity of approximately solving semidefinite programs in the bit model; see, e.g., [GM12, Thm. 2.6.1] for a concrete formulation based on general theorems of [GLS88]. However, the main difficulty one faces when trying to apply such a bound to polynomial optimization is that the ellipsoid algorithm, in order to be guaranteed to find a feasible solution, needs that the set of feasible solutions be suitably bounded (which can be arranged in our setting) and contains an  $\varepsilon$ -ball, for  $\varepsilon > 0$  with polynomially many bits. (The ball is not in the space of *all* positive semidefinite matrices, but rather in the space of all such matrices satisfying all equality constraints of the semidefinite program.) The latter condition, for semidefinite programs coming from polynomial optimization problems, looks at least non-obvious, and perhaps it might even fail in some cases.

We believe that this kind of theory is worth working out, preferably in the general context of multivariate polynomial optimization as in [Las10]—at least we could not find any study in this direction.

## 2 Preliminaries

**Divided differences and  $k$ -monotonicity.** The  $k$ th divided difference of a real function  $f$  at points  $x_0, x_1, \dots, x_k \in \mathbb{R}$  is denoted by  $[x_0, x_1, \dots, x_k]f$  and defined recursively by

$$[x_0]f := f(x_0), \quad [x_0, \dots, x_k]f := \frac{[x_1, \dots, x_k]f - [x_0, \dots, x_{k-1}]f}{x_k - x_0}.$$

It is known that  $f$  is  $k$ -monotone on an open interval  $I$  iff  $[x_0, x_1, \dots, x_k]f \geq 0$  for all choices of  $x_0 < x_1 < \dots < x_k \in I$  (see [KS03, Lemma 3.1]).

Sometimes it will be notationally convenient to regard a set  $P$  of points in the plane with distinct  $x$ -coordinates as the graph of a function  $f: X \rightarrow \mathbb{R}$ , where  $X = X(P)$  is the set of the  $x$ -coordinates of the points of  $P$ . Then, instead of  $P$  being  $k$ -monotone interpolable, we can also say that  $(X, f)$  is  $k$ -monotone interpolable.

Here is a useful criterion for determining the sign of the divided difference  $[x_0, x_1, \dots, x_k]f$ , where  $x_0 < x_1 < \dots < x_k$ : Let  $i \in \{0, 1, \dots, k\}$ , and let  $p$  be the unique polynomial of degree at most  $k-1$  such that  $p(x_j) = f(x_j)$  for all  $j \in \{0, 1, \dots, k\} \setminus \{i\}$ . Then  $\text{sgn}[x_0, x_1, \dots, x_k]f = (-1)^{k-i} \text{sgn}(p(x_i) - f(x_i))$  (see [EM13]). So, for example, for  $k = 3$ , if we pass a parabola through the first three values of  $f$ , then the fourth value is above the parabola for  $[x_0, x_1, x_2, x_3]f > 0$ , and below it for  $[x_0, x_1, x_2, x_3]f < 0$ .

A necessary condition for  $k$ -monotone interpolability of  $(X, f)$  is  $[x_0, \dots, x_k]f \geq 0$  for every choice of  $x_0 < x_1 < \dots < x_k \in X$ . While, as was discussed in the introduction, this condition is very far from sufficient for arbitrary  $X$ , it is sufficient for  $|X| = k + 1$  (e.g., because  $[x_0, \dots, x_k]f$  is the leading coefficient of the unique polynomial  $p$  of degree at most  $k$  that coincides with  $f$  on  $X$ , and if this coefficient is nonnegative, then  $p$  is a  $k$ -monotone interpolant; see, e.g., [EM13]).

**A representation theorem for  $k$ -monotone functions.** The following characterization of  $k$ -monotone function essentially goes back to Schoenberg [Sch41]; see [KS03].

**Theorem 2.1** (Representation theorem). *A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is  $k$ -monotone if and only if for every closed interval  $[a, b]$  there is a polynomial  $p(x)$  of degree at most  $k - 1$  and a bounded nondecreasing function  $\mu: [a, b] \rightarrow \mathbb{R}$  such that*

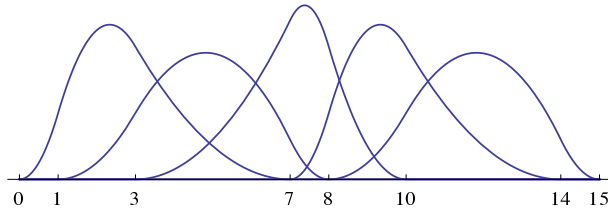
$$f(x) = p(x) + \frac{1}{k!} \int_a^b k \max(x - t, 0)^{k-1} d\mu(t), \quad x \in [a, b].$$

This basically says that a  $k$ -monotone function must be a nonnegative linear combination of translates of the function  $\max(x, 0)^{k-1}$ , plus a polynomial of degree at most  $k - 1$  (except that we do not have a finite linear combination but an integral). In particular, a 3-monotone function can be made of a parabola and “right half-parabolas”.

**A characterization of  $k$ -monotone interpolability.** Let  $X = \{x_1, \dots, x_{n+k}\} \subset \mathbb{R}$ ,  $x_1 < x_2 < \dots < x_{n+k}$ , be a set of  $n+k$  real numbers, which are often referred to as *nodes* in this context. The *B-splines* of degree  $k - 1$  corresponding to  $X$  are the functions  $M_1(t), \dots, M_n(t)$  defined by the formula

$$M_i(t) := k[x_i, \dots, x_{i+k}] \max(0, x - t)^{k-1},$$

where the divided differencing on the right-hand side is with respect to  $x$  (while  $t$  is viewed as a fixed parameter). Here is an example with  $k = 3$  (the nodes are marked on the  $x$ -axis, and the peaks of  $M_1, \dots, M_5$  go in the left-to-right order):



Each  $M_i$  is strictly positive on the interval  $(x_i, x_{i+k})$  and zero outside of it, and on each interval  $[x_j, x_{j+1}]$ , each  $M_i$  equals some polynomial  $p_{ij}$  of degree at most  $k - 1$ .

The characterization of  $k$ -monotone interpolability we will use was obtained from Theorem 2.1 by a duality argument, and it can be stated as follows.

**Lemma 2.2** ([KS03], Corollary 6.5). *Let  $X = \{x_1, \dots, x_{n+k}\}$ ,  $x_1 < \dots < x_{n+k}$ , be a node sequence, let  $f: X \rightarrow \mathbb{R}$  be a function, and let the vector  $v = (v_1, \dots, v_n)$  be given by  $v_i = [x_i, \dots, x_{i+k}]f$ . Then  $(X, f)$  is  $k$ -monotone interpolable if and only if the following implication holds for every  $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ : If  $\sum_{i=1}^n a_i M_i(t) \geq 0$  for all  $t \in [x_1, x_{n+k}]$ , then  $\sum_{i=1}^n a_i v_i \geq 0$ .*

Geometrically, if we denote by  $\mathbf{M}$  the compact set

$$\mathbf{M} = \left\{ (M_1(t), \dots, M_n(t)) \in \mathbb{R}^n : t \in [x_1, x_{n+k}] \right\},$$

then the characterization says that  $P$  is not  $k$ -monotone interpolable if and only if the point  $v$  can be strictly separated from  $\mathbf{M}$  by a hyperplane passing through the origin.

### 3 Proof of Theorem 1.2 (Ramsey-type result)

The following alternative criterion for  $k$ -monotone interpolability can be derived from the representation theorem (Theorem 2.1) or from Lemma 2.2.

**Lemma 3.1.** *Let  $X = \{x_1, \dots, x_{n+k}\}$ ,  $x_1 < \dots < x_{n+k}$ , be a node sequence, let  $f: X \rightarrow \mathbb{R}$  be a function, and let the vector  $v = (v_1, \dots, v_n)$  be given by  $v_i = [x_i, \dots, x_{i+k}]f$ . Then  $(X, f)$  is  $k$ -monotone interpolable if and only if there exist  $c_1, \dots, c_n \geq 0$  and  $t_1, \dots, t_n \in [x_1, x_{n+k}]$  satisfying  $v_i = \sum_{j=1}^n c_j M_i(t_j)$  for all  $i = 1, \dots, n$ .*

*Proof.* The “if” part is obvious from Lemma 2.2: the condition guarantees that  $v$  lies in the convex cone generated by the set  $\mathbf{M}$  defined after Lemma 2.2, and hence it cannot be separated from  $\mathbf{M}$ .

The “only if” part follows from a suitable hyperplane separation theorem for convex cones; one needs to verify that the cone generated by  $\mathbf{M}$  is closed. We omit the details since we do not need the “only if” direction.  $\square$

We are now ready to prove the Ramsey-type result.

*Proof Theorem 1.2.* Let  $P = \{(x, f(x)) : x \in X\} \subset \mathbb{R}^2$  be an  $N$ -point set with distinct  $x$ -coordinates.

A necessary condition for 3-monotone interpolability of  $P$  is that  $[x_0, \dots, x_3]f \geq 0$  for every choice of  $x_0 < \dots < x_3 \in X$ . This condition can be easily enforced using Ramsey’s theorem for fourtuples: we color a fourtuple  $\{x_0, \dots, x_3\} \subseteq X$  red if  $[x_0, \dots, x_3]f \geq 0$  and blue otherwise, and if  $N$  is sufficiently large, we can select a subset  $Y \subseteq X$  of prescribed size in which all fourtuples have the same color. By possibly passing to  $P^\uparrow$ , we may thus assume that  $[x_0, \dots, x_3]f \geq 0$  for all fourtuples in  $Y$ .

Next, by Ramsey’s theorem again, we will select an  $(n+3)$ -point subset  $Z = \{z_1, \dots, z_{n+3}\} \subseteq Y$ , which we will prove to be 3-monotone interpolable. This time we will 2-color 5-tuples, in a way which looks mysterious at first sight, but which will be explained by the proof below.

For a node sequence  $U = \{u_1 < \dots < u_{m+3}\}$  of real numbers, let  $M_i^U$  be  $i$ th B-spline of degree 2, i.e.,  $[u_i, u_{i+1}, u_{i+2}, u_{i+3}] \max(0, x - t)^2$ . For  $U \subseteq Y$ , we



also write  $v_i^U$  for  $[u_i, u_{i+1}, u_{i+2}, u_{i+3}]f$ . Note that our choice of  $Y$  guarantees  $v_i^U \geq 0$  for every  $U \subseteq Y$  and all  $i$ .

Now we define the 2-coloring of the 5-tuples: a 5-tuple  $U = \{u_1 < \dots < u_5\} \subseteq Y$  is *v-positive* if

$$\frac{v_1^U}{M_1^U(u_3)} \leq \frac{v_2^U}{M_2^U(u_3)},$$

and otherwise it is *v-negative*.

We recall that  $M_i^U(u_j)$  is strictly positive for  $j = i + 1$  and  $j = i + 2$  and zero for all other  $j$ , and so the coloring is well defined.

By Ramsey's theorem, if  $|Y|$  is sufficiently large, there exists  $Z = \{z_1 < \dots < z_{n+3}\} \subseteq Y$  with all 5-tuples of the same type (i.e. either all v-positive or all v-negative). We will use Lemma 3.1 with  $X = Z$  to show that  $Z$  is 3-monotone interpolable. From now until the end of the proof, to simplify the notation, let us write  $M_i$  for  $M_i^Z$  and  $v_i$  for  $v_i^Z$ .

**The v-positive case.** Here we choose  $t_j := z_{j+2}$ ,  $j = 1, \dots, n$ , in Lemma 3.1. With the  $t_j$  fixed, the conditions  $v_i = \sum_{j=1}^n c_j M_i(t_j)$  provide a system of  $n$  linear equations for the unknowns  $c_1, \dots, c_n$ .

The idea is to calculate  $c_1$ , then  $c_2$ , then  $c_3$ , etc. from these linear equations. In the  $i$  step,  $i \geq 2$ , v-positivity is exactly the right condition for ensuring that  $c_i \geq 0$ .

Since  $M_i(z_{j+2})$  is zero unless  $j \in \{i - 1, i\}$ , the first equation reads  $v_1 = c_1 M_1(z_3)$  and determines  $c_1 = v_1 / M_1(z_3)$  uniquely. We also have  $c_1 \geq 0$  since  $v_i \geq 0$ , by the choice of  $Y$ .

Now we suppose inductively that nonnegative  $c_1, \dots, c_i$  have been determined, in such a way that they satisfy the first  $i$  equations. Moreover, to support the induction, we also assume  $c_i \leq v_i / M_i(z_{i+2})$ .

Then expressing  $c_{i+1}$  from the  $(i + 1)$ st equation gives

$$c_{i+1} := \frac{v_{i+1} - c_i M_{i+1}(z_{i+2})}{M_{i+1}(z_{i+3})}.$$

Since  $c_i \geq 0$  and  $M_{i+1} \geq 0$ , this formula implies the inequality  $c_{i+1} \leq v_{i+1} / M_{i+1}(z_{i+3})$  needed for our induction.

It remains to verify that  $c_{i+1} \geq 0$ , and here we use the v-positivity of the 5-tuple  $\{z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4}\}$ , from which we obtain

$$c_i \leq \frac{v_i}{M_i(z_{i+2})} \leq \frac{v_{i+1}}{M_{i+1}(z_{i+2})}.$$

Hence the numerator in the formula for  $c_{i+1}$  is nonnegative. This finishes the inductive step; we have shown that the condition in Lemma 3.1 is fulfilled and so the restriction of  $f$  to  $Z$  is 3-monotone interpolable.

**The v-negative case.** This case is similar to the previous one, but this time we set  $t_j := z_{j+1}$  (as opposed to  $t_j = z_{j+2}$  in the previous case), and we work backwards, computing first  $c_n$ , then  $c_{n-1}$ , etc.

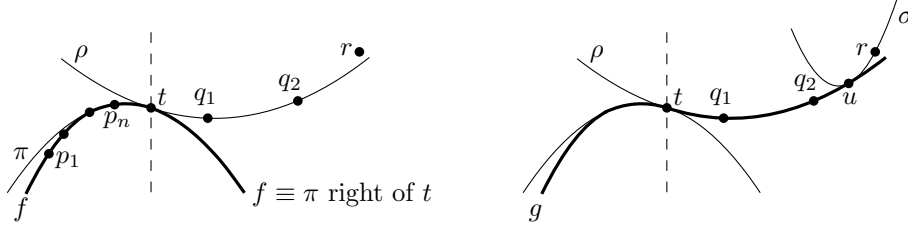


Figure 1: Proof of Lemma 4.1.

From the  $n$ th equation we obtain  $c_n = v_n/M_n(z_{n+1})$ . In the inductive step, we assume that nonnegative  $c_n, \dots, c_{i+1}$  have been determined satisfying the last  $n - i$  equations and such that  $c_{i+1} \leq v_{i+1}/M_{i+1}(z_{i+2})$ . Then the  $i$ th equation dictates that

$$c_i := \frac{v_i - c_{i+1}M_i(z_{i+2})}{M_i(z_{i+1})}.$$

As before,  $c_i \leq v_i/M_i(z_{i+1})$  follows immediately. The  $v$ -negativity of  $\{z_i, \dots, z_{i+4}\}$  then yields

$$c_{i+1} \leq \frac{v_{i+1}}{M_{i+1}(z_{i+2})} < \frac{v_i}{M_i(z_{i+2})},$$

again showing the numerator in the formula for  $c_i$  nonnegative. This concludes the proof.  $\square$

## 4 Constructions of point sets

We are going to prove Theorems 1.3 and 1.4. The idea of both constructions is similar, and first we prepare a result common for both of them. But while it is possible to arrange the construction for Theorem 1.4 so that it also verifies Theorem 1.3, the technical details come out complicated, and so we prefer to keep the two constructions separate.

For a point  $p \in \mathbb{R}$ , we write  $x(p)$  and  $y(p)$  for the  $x$  and  $y$  coordinates of  $p$ .

**Lemma 4.1.** *Let  $P = \{p_1, \dots, p_n\}$  be a 3-monotone interpolable point set where  $x(p_1) < \dots < x(p_n)$ . Assume that for some parabola  $\pi$ , there is a 3-monotone interpolant  $f$  of  $P$  equal to  $\pi$  to the right of  $p_n$ . Also assume that for a point  $q$  to the right of  $p_n$ ,  $P \cup \{q\}$  is 3-monotone interpolable if and only if  $q$  lies on or above  $\pi$ . Further, let  $Q = \{q_1, q_2\}$  be a pair of points above  $\pi$  that satisfy  $x(p_n) < x(q_1) < x(q_2)$  and such that there is a parabola  $\rho$  passing through  $q_1$  and  $q_2$  tangent to  $\pi$  with the point of tangency to the right of  $p_n$  and to the left of  $q_1$  (see Fig. 1 left).*

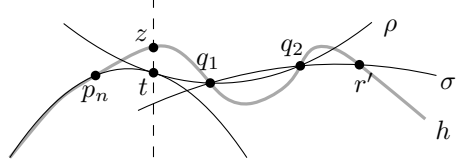
*Then for a point  $r$  to the right of  $q_2$ ,  $P \cup Q \cup \{r\}$  is 3-monotone interpolable if and only if  $r$  lies on  $\rho$  or above it.*

*Proof.* Let  $t$  be the point of tangency of  $\pi$  and  $\rho$ . Notice that the curve  $g$  equal to  $f$  to the left of  $t$  and equal to  $\rho$  to the right of  $t$  is a 3-monotone interpolant of

$P \cup Q$ ; see Fig. 1. Consequently, if  $r$  lies on  $\rho$  to the right of  $q_2$ , then  $P \cup Q \cup \{r\}$  is 3-monotone interpolable.

Now assume that  $r$  lies to the right of  $q_2$  and above  $\rho$ . For every point  $u$  on  $\rho$  with  $x(u) \neq x(r)$ , there is a (unique) parabola passing through  $r$  that is tangent to  $\rho$  with  $u$  as the point of tangency. We fix a point  $u$  on  $\rho$  with  $x(q_2) < x(u) < x(r)$  and a parabola  $\sigma$  passing through  $r$  and tangent to  $\rho$  in  $u$ . The curve equal to  $g$  to the left of  $u$  and equal to  $\sigma$  to the right of  $u$  is a 3-monotone interpolant of  $P \cup Q \cup \{r\}$ .

Now we consider  $r'$  to the right of  $q_2$  and below  $\rho$ . We assume, for contradiction, that  $P \cup Q \cup \{r'\}$  has a 3-monotone interpolant  $h$ . Let  $\sigma'$  be the parabola containing  $q_1$ ,  $q_2$  and  $r'$ :



Then  $\sigma'$  and  $\rho$  have exactly two points in common:  $q_1$  and  $q_2$ . Therefore  $\sigma'$  is strictly below  $\rho$  everywhere to the left of  $q_1$ . We consider the point  $z = (x(t), h(x(t)))$ . For the quadruple  $\{z, q_1, q_2, r'\}$  to be positive,  $z$  has to lie on  $\sigma'$  or below it. On the other hand, since  $h$  is a 3-monotone interpolant of  $P \cup \{z\}$ ,  $z$  lies on or above  $\pi$ . This is a contradiction, since  $\pi(x(t)) = \rho(x(t)) > \sigma'(x(t))$ .  $\square$

Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be two functions. A *convex combination* of  $f$  and  $g$  is the function  $\alpha f + (1 - \alpha)g$  for some  $\alpha \in [0, 1]$ .

**Observation 4.2.** *Let  $k \geq 1$ . Let  $f$  and  $g$  be two  $k$ -monotone interpolants of a set  $P$ . Then every convex combination of  $f$  and  $g$  is a  $k$ -monotone interpolant of  $P$ .*

#### 4.1 Proof of Theorem 1.3 (non-locality)

We will prove the following by induction on  $i$ :

**Claim 4.3.** *For every  $i \geq 1$  there exists a set  $P_i$  of  $2i + 1$  points in the plane and an integer  $u_i$  that satisfy the following. There are quadratic functions  $\pi_i$  and  $\pi_{i,p}$ ,  $p \in P_i$ , where each  $\pi_{i,p}(x) \leq \pi_i(x) - 1$  on  $[u_i, \infty)$  for every  $p \in P_i$ , such that:*

- (i) *There exists a 3-monotone interpolant  $f$  for  $P_i$  that equals  $\pi_i$  on  $[u_i, \infty)$ , but if  $q$  is a point with  $x(q) \geq u_i$  and strictly below  $\pi_i$ , then  $P_i \cup \{q\}$  is not 3-monotone interpolable.*
- (ii) *For every  $p \in P_i$ , the set  $P_i \setminus \{p\}$  is 3-monotone interpolable, and among the 3-monotone interpolants, there is a function  $f_{i,p}$  that equals  $\pi_{i,p}$  on  $[u_i, \infty)$ .*

Moreover, the coordinates of all the points in  $P_i$  are integers from the range  $0, \dots, 25i^3$  and  $\pi_i(u_i)$  is an integer.

*Proof.* Define  $u_i = 5i$ .

When  $i = 1$ , the requirements are satisfied by the triple of points  $(0, 0)$ ,  $(1, 0)$ ,  $(2, 0)$ .

For  $i \geq 2$ , we proceed by induction. We have a set  $P_{i-1}$  of  $2i - 1$  points and quadratic functions  $\pi_{i-1}$  and  $\pi_{i-1,p}$  for every  $p \in P_{i-1}$ .

We define

$$\pi_i(x) = \pi_{i-1}(x) + (x - u_{i-1} - 1)^2.$$

Thus,  $\pi_i$  is a parabola tangent to  $\pi_{i-1}$  at a point with  $x$ -coordinate  $u_{i-1} + 1$ . We also have  $\pi_i(x) > \pi_{i-1}(x)$  for every  $x \in \mathbb{R} \setminus \{u_{i-1} + 1\}$ . We now define the set  $P_i$  as  $P_i = P_{i-1} \cup \{p_{2i}, p_{2i+1}\}$ , where  $p_{2i}$  and  $p_{2i+1}$  are points on  $\pi_i$  with  $x$ -coordinates  $u_{i-1} + 2$  and  $u_{i-1} + 3$ .

Claim (i) follows from Lemma 4.1.

Now we verify claim (ii).

If  $p \in P_{i-1}$ , we consider the 3-monotone interpolant  $f_{i-1,p}$  of  $P \setminus \{p\}$  that equals  $\pi_{i-1,p}$  on  $[u_{i-1}, \infty)$ . We define the parabola  $\pi_{i,p}$  as the parabola that passes through  $(u_{i-1}, \pi_{i-1}(u_{i-1}))$ ,  $p_{2i}$  and  $p_{2i+1}$ . That is,  $\pi_i(x) - \pi_{i,p}(x)$  is a quadratic function that attains its minimum at  $u_{i-1} + 5/2$  and is equal to 1 at  $u_{i-1}$ . Then we have  $\pi_{i,p}(x) \leq \pi_i(x) - 1$  on  $[u_i, \infty)$ . We also deduce

$$\pi_{i,p}(x) = \pi_{i-1}(x) + \frac{5}{6}(x - u_{i-1})^2 - \frac{7}{6}(x - u_{i-1}).$$

So we have  $\pi_{i,p}(x) > \pi_{i-1}(x) - 1 \geq \pi_{i-1,p}(x)$  for every  $x \in [u_{i-1}, \infty)$ .

Since  $\pi_{i,p}$  has two intersections with  $f_{i-1}$  and no intersection with  $f_{i-1,p}$  on  $[u_i, \infty)$ ,  $f_{i-1}$  and  $f_{i-1,p}$  have a convex combination  $g$  whose restriction on  $[u_{i-1}, \infty)$  is a parabola tangent to  $\pi_{i,p}$ . By Observation 4.2,  $g$  is a 3-monotone interpolant of  $P \setminus \{p\}$ .

Let  $t'$  be the point of tangency. The function  $f_{i,p}$  equal to  $g$  on  $(-\infty, x(t'))$  and equal to  $\pi_{i,p}$  on  $[x(t'), \infty)$  is a 3-monotone interpolant for  $P_i \setminus \{p\}$  that equals  $\pi_{i,p}$  on  $[u_i, \infty)$ .

If  $p = p_{2i}$  or  $p = p_{2i+1}$ , we let  $q$  be the point from  $\{p_{2i}, p_{2i+1}\}$  different from  $p$ . We take the parabola  $\pi_{i,p}$  that passes through  $q$  and is tangent to  $\pi_{i-1}$  at a point  $t'$  with  $x(t') = u_{i-1}$ . We have

$$\begin{aligned} \pi_{i,p}(x) &= \pi_{i-1}(x) + \frac{4}{9}(x - u_{i-1})^2 \text{ when } p = p_{2i} \text{ and} \\ \pi_{i,p}(x) &= \pi_{i-1}(x) + \frac{1}{4}(x - u_{i-1})^2 \text{ when } p = p_{2i+1}. \end{aligned}$$

In both cases,  $\pi_{i,p}(x) \leq \pi_i(x) - 1$  on  $[u_i, \infty)$ . The function  $f_{i,p}$  equal to  $f_{i-1}$  on  $(-\infty, u_{i-1}]$  and equal to  $\pi_{i,p}$  on  $[u_{i-1}, \infty)$  is a 3-monotone interpolant for  $P_i \setminus \{p\}$  that satisfies claim (ii).

The  $x$ -coordinates of all the points of  $P_i$  are integers from  $\{0, \dots, 5i\}$  and lie on the parabolas  $\pi_i$ . All coefficients of the quadratic functions  $\pi_i$  are integers and so all the points in  $P_i$  have integer coordinates. We have  $\pi_1 \equiv 0$  and for every integer  $i$  and every real  $x \in [0, 5i]$ , we have  $\pi_i(x) \leq \pi_{i-1}(x) + (5i)^2$  and so  $\pi_i(x) \leq 25i^3$ .  $\square$

We are now ready to prove Theorem 1.3. We have  $n = 2i + 2$  for some  $i \geq 1$ . The set  $P$  is formed by all the points of  $P_i$  and a point  $q = (u_i, \pi_i(u_i) - 1)$ .

## 4.2 Proof of Theorem 1.4 (doubly exponentially small example)

**Lemma 4.4.** *Let  $\varepsilon > 0$ . For every  $j$ , let  $p_j$  be the point  $(j, j^3)$  and let  $q_j$  be the point  $(j, j^3 + \varepsilon)$ . Given an arbitrary integer  $i$ , let  $\kappa$  be the parabola passing through  $q_{i-2}$ ,  $p_{i-1}$  and  $p_i$ . Then  $(i+1)^3 - \kappa(i+1) = 6 - \varepsilon$  and  $(i+2)^3 - \kappa(i+2) = 24 - 3\varepsilon$ .*

*Proof.* We first consider the parabola  $\tau_i$  passing through  $p_i$ ,  $p_{i+1}$  and  $p_{i+2}$  and a parabola  $\tau_{i-2}$  passing through  $p_{i-2}$ ,  $p_{i-1}$  and  $p_i$ . By a straightforward calculation, for every  $x \in \mathbb{R}$ ,

$$\tau_i(x) = (3i + 3)x^2 - (3i^2 + 6i + 2)x + i^3 + 3i^2 + 2i$$

and

$$\tau_i(x) - \tau_{i-2}(x) = 6(x - i)^2.$$

Let  $\delta$  be the quadratic function  $\kappa - \tau_{i-2}$ . We have  $\delta(i - 2) = \varepsilon$ ,  $\delta(i - 1) = 0$  and  $\delta(i) = 0$ . Thus, for every  $x \in \mathbb{R}$ :

$$\kappa(x) - \tau_{i-2}(x) = \delta(x) = \frac{\varepsilon}{2} \cdot (x - i + 1/2)^2 - \varepsilon/8.$$

It is now easy to calculate the values  $\tau_i(x) - \kappa(x)$  for  $x = i + 1$  and  $x = i + 2$  and verify the claim.  $\square$

**Lemma 4.5.** *For every  $j$ , let  $p_j$  be the point  $(j, j^3)$ . For an integer  $i > 2$  and an arbitrary  $\varepsilon \in (0, 1]$ , let  $q_{i+1}$  be the point  $(i + 1, (i + 1)^3 - 6 + \varepsilon)$ . Let  $\tau$  be the parabola passing through  $p_{i-1}$ ,  $p_i$  and  $q_{i+1}$ . Then there is a parabola  $\pi$  passing through  $p_{i+1}$  and  $p_{i+2}$  that is tangent to  $\tau$  such that the  $x$ -coordinate of the point of tangency is in the interval  $(i, i + 1)$ . Moreover,  $\pi(i + 3) = (i + 3)^3 - 6 + \delta$ , where  $\delta \in (0, \varepsilon^2/5)$ .*

*Proof.* Let  $p'_{i+1} = (1, (i + 1)^3 - \tau(i + 1))$  and  $p'_{i+2} = (2, (i + 2)^3 - \tau(i + 2))$ . From Lemma 4.4, we have  $p'_{i+1} = (1, 6 - \varepsilon)$  and  $p'_{i+2} = (1, 24 - 3\varepsilon)$ . The main part of the proof is finding a parabola  $\pi'$  passing through  $p'_{i+1}$  and  $p'_{i+2}$  that is tangent to the  $x$ -axis in a point with  $x$ -coordinate in  $(0, 1)$ . Then we show that the parabola  $\pi$  defined by  $\pi(x) = \pi'(x - i) + \tau(x)$  for every  $x \in \mathbb{R}$  has the claimed properties.

Since  $p'_{i+2}$  is higher than  $p'_{i+1}$  and both are above the  $x$ -axis, there are exactly two parabolas passing through  $p'_{i+1}$  and  $p'_{i+2}$  that are tangent to the  $x$ -axis. The point of tangency of one of the two parabolas is between  $p'_{i+1}$  and  $p'_{i+2}$ , while the point of tangency of the other is to the left of  $p'_{i+1}$ . The parabola with tangency to the left of  $p'_{i+1}$  goes below the other parabola everywhere to the left of  $p'_{i+1}$  and thus has a smaller coefficient of the quadratic term.

We write  $\pi'(x) = ax^2 + bx + c$ . Since  $\pi'$  passes through  $p'_{i+1}$  and  $p'_{i+2}$  and is tangent to the  $x$ -axis, we have

$$\begin{aligned} a + b + c &= 6 - \varepsilon \\ 4a + 2b + c &= 24 - 3\varepsilon \\ b^2 &= 4ac. \end{aligned}$$

To simplify the equations, we define  $\bar{a} = a - 6$ . Using the first two equations, we express  $b$  and  $c$  in terms of  $\bar{a}$  and  $\varepsilon$  as  $b = -3\bar{a} - 2\varepsilon$  and  $c = 2\bar{a} + \varepsilon$ . The third equation then becomes

$$\bar{a}^2 + 8\bar{a}\varepsilon - 48\bar{a} + 4\varepsilon^2 - 24\varepsilon = 0.$$

Let  $f(\bar{a})$  be the left-hand side of the equation. Using  $\varepsilon \in (0, 1]$ , it is easy to verify that  $f(-\varepsilon/2) > 0$ ,  $f(0) < 0$  and that  $f$  goes to infinity as  $\bar{a}$  goes to infinity. Let  $\bar{a}_1$  and  $\bar{a}_2$  be the two roots of  $f(\bar{a})$  with  $\bar{a}_1 < \bar{a}_2$ . Since the value of  $\bar{a}$  corresponding to the parabola  $\pi'$  is the smaller of the two roots of  $f(\bar{a})$ , its value is  $\bar{a}_1 \in (-\varepsilon/2, 0)$ . We then have  $a \in (5, 6)$  and  $b \in (-2\varepsilon, -\varepsilon/2)$ .

The  $x$ -coordinate of the point of tangency of  $\pi'$  with the  $x$ -axis is

$$\frac{-b}{2a} \in \left(0, \frac{\varepsilon}{5}\right) \subset (0, 1).$$

From  $b^2 = 4ac$ , we obtain

$$c = \frac{b^2}{4a} \in \left(0, \frac{\varepsilon^2}{5}\right).$$

We define  $\delta = c$ . Notice that  $\pi'$  passes through the point  $(0, \delta)$ .

Consequently, the parabola  $\pi$  passes through  $p_{i+1}$  and  $p_{i+2}$  and is tangent to  $\tau$  in a point with  $x$ -coordinate in the interval  $(i, i+1)$  and passes through the point  $(i, i^3 + \delta)$ . By Lemma 4.4,  $\pi(i+3) = (i+3)^3 - 6 + \delta$ .  $\square$

The next lemma is a slight strengthening of Theorem 1.4.

**Lemma 4.6.** *Let  $p_j$  be the point  $(j, j^3)$  and let  $z = (-1, 0)$ . Let  $P_m = \{z, p_0, p_1, \dots, p_{2m+1}\}$ . For every integer  $m \geq 0$ , we consider the point  $q_{2m+2}$  with  $x$ -coordinate  $2m+2$  and with the smallest possible  $y$ -coordinate such that the set  $P_m \cup \{q_{2m+2}\}$  is 3-monotone interpolable. Then the  $y$ -coordinate of  $q_{2m+2}$  equals  $(2m+2)^3 - 6 + \varepsilon_m$  for some positive  $\varepsilon_m \leq 2 \cdot 2^{-2^m}$ .*

*Proof.* Let  $\pi_0$  be the parabola passing through  $z$ ,  $p_0$  and  $p_1$ . Observe that  $\pi_0(2) = 3$  and thus the claim holds for  $m = 0$  with  $\varepsilon_0 = 1$ .

We now consider the inductive step for  $m \geq 1$ .

Let  $\pi_{m-1}$  be the parabola passing through  $p_{2m-2}$ ,  $p_{2m-1}$  and  $q_{2m}$ . As a consequence of the induction hypothesis, for every point  $s$  to the right of  $p_{2m-1}$ ,  $P_{m-1} \cup \{s\}$  is 3-monotone interpolable if and only if  $s$  lies on or above  $\pi_{m-1}$ .

By Lemma 4.5, there is a parabola  $\pi_m$  passing through  $p_{2m}$  and  $p_{2m+1}$  that is tangent to  $\pi_{m-1}$  in a point to the left of  $p_{2m}$  and to the right of  $p_{2m-1}$ .

By Lemma 4.1, for every point  $s$  to the right of  $p_{2m+1}$ ,  $P_m \cup \{s\}$  is 3-monotone interpolable if and only if  $s$  lies above  $\pi_m$ . By Lemma 4.5,  $\pi_m(2m+2) = (2m+2)^3 - 6 + \varepsilon_m$  for some  $\varepsilon_m \in (0, \varepsilon_{m-1}^2/5)$ . That is,

$$\varepsilon_m \leq \frac{\varepsilon_{m-1}^2}{5} \leq \frac{(2 \cdot 2^{-2^{m-1}})^2}{5} \leq \frac{4}{5} \cdot 2^{2 \cdot (-2^{m-1})} \leq 2 \cdot 2^{-2^m}.$$

$\square$

## 5 Proof of Theorem 1.4 (exponentially many digits)

### 5.1 The semidefinite formulation

By the characterization in Lemma 2.2, if we think of a point set  $P \subset \mathbb{R}^2$  as a function  $f: X \rightarrow \mathbb{R}$ , with  $X = \{x_1, \dots, x_{n+k}\}$ , then  $(X, f)$  is *not*  $k$ -monotone interpolable exactly if there is  $a \in \mathbb{R}^n$  such that  $\sum_{i=1}^n a_i M_i(t) \geq 0$  for all  $t \in [x_1, x_{n+k}]$  and  $\sum_{i=1}^n a_i v_i = -1$ , where the  $v_i = [x_i, \dots, x_{i+k}]f$  are the  $k$ th divided differences. Further we recall that  $M_i(t)$  equals a polynomial  $p_{ij}(t)$  of degree at most  $k-1$  on each interval  $[x_j, x_{j+1}]$ .

By re-scaling the interval  $[x_j, x_{j+1}]$  to  $[-1, 1]$  for notational convenience, each  $p_{ij}$  is transformed into another polynomial  $\tilde{p}_{ij}$ . The coefficients of  $\tilde{p}_{ij}$  can obviously be computed from the  $x_i$  in polynomial time. Thus, the impossibility of  $k$ -monotone interpolation is a special case of the following computational problem.

**Problem 5.1** (The non-positivity<sup>2</sup> problem).

Input: Polynomials  $\tilde{p}_{ij}(t)$ ,  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, m$  with rational coefficients and a vector  $v \in \mathbb{Q}^n$ .

Question: Does there exist  $a \in \mathbb{R}^n$  such that  $\sum_{i=1}^n a_i \tilde{p}_{ij}(t) \geq 0$  for all  $t \in [-1, 1]$  and all  $j = 1, \dots, m$ , and  $\sum_{i=1}^n a_i v_i = -1$ ?

There is a large body of work showing that problems involving nonnegativity of polynomials over semialgebraic sets (i.e., sets defined by polynomial inequalities) can be converted, under fairly general conditions, to semidefinite programs.

**Semidefinite programs.** We recall that a *semidefinite program* is the computational problem of finding a positive semidefinite  $n \times n$  matrix  $X$  that maximizes a linear function  $C \bullet X$  subject to linear constraints  $A_1 \bullet X = b_1, \dots, A_2 \bullet X = b_m$ , for given  $n \times n$  matrices  $C$  and  $A_1, \dots, A_m$  and reals  $b_1, \dots, b_m$ . Here the matrix scalar product  $\bullet$  is defined as  $C \bullet X = \sum_{i,j=1}^n c_{ij} x_{ij}$ . We refer, e.g., to the books [BTN01, BV04, GM12] or handbooks [WSV00, AL12] for background.

For the semidefinite formulation of our non-positivity problem, the maximized function  $C \bullet X$  is irrelevant; we need only the *semidefinite feasibility problem*, where we ask for the existence of a positive semidefinite  $X$  satisfying given linear constraints.

**Semidefinite formulation of the non-positivity problem.** By a classical result, see [Las10, Theorem 2.6], a univariate polynomial  $p(t)$  of degree  $d$  is nonnegative on  $[-1, 1]$  if and only if it can be written as

$$p(t) = f(t) + (1-t)(1+t)h(t),$$

where  $f(t)$  and  $h(t)$  are polynomials that can be expressed as sums of squares of suitable polynomials, i.e., in the form  $\sum_{i=1}^m s_i(t)^2$  for some  $m$  and some polynomials  $s_1(t), \dots, s_m(t)$ , with  $\deg f \leq 2d$  and  $\deg h \leq 2d-2$ .

<sup>2</sup>The word positivity refers to a customary terminology: a vector  $v$  is called *positive* w.r.t. a system  $u_1, \dots, u_n$  of real functions on an interval  $I$  if  $\sum_{i=1}^n a_i u_i(t) \geq 0$  for all  $t \in I$  implies  $\sum_{i=1}^n a_i v_i \geq 0$ .

Moreover, a polynomial  $f(t)$  is a sum of squares of degree at most  $2d$  iff it has the form  $\mathbf{t}^T Q \mathbf{t}$ , where  $Q$  is a  $(d+1) \times (d+1)$  positive semidefinite matrix and  $\mathbf{t} = (1, t, t^2, \dots, t^d)$ ; see [Las10, Prop. 2.1].

Thus, a polynomial  $p(t)$  of degree at most  $d$  is nonnegative on  $[-1, 1]$  if and only if there are a  $(d+1) \times (d+1)$  matrix  $Q$  and  $d \times d$  matrix  $\tilde{Q}$ , both positive semidefinite, such that

$$p(t) = \mathbf{t}^T Q \mathbf{t} + (1-t)(1+t)\tilde{\mathbf{t}}^T \tilde{Q} \tilde{\mathbf{t}}$$

holds as equality of polynomials in  $t$ , where  $\tilde{\mathbf{t}} = (1, t, \dots, t^{d-1})$ . Expanding each side according to powers of  $t$ , we obtain  $2d+1$  linear equations involving the entries of  $Q$  and  $\tilde{Q}$  and the coefficients of  $p(t)$ .

Therefore, the non-positivity problem above can be re-stated as the existence of reals  $a_1, \dots, a_n$  and positive semidefinite matrices  $Q_1, \dots, Q_m$  (of size  $k \times k$ ) and  $\tilde{Q}_1, \dots, \tilde{Q}_m$  (of size  $(k-1) \times (k-1)$ ) such that  $\sum_{i=1}^n a_i v_i = -1$  and for each  $j = 1, 2, \dots, m$ , the matrices  $Q_j$  and  $\tilde{Q}_j$  witness the nonnegativity of the polynomial  $\sum_{i=1}^n a_i \tilde{p}_{ij}(t)$  in the above sense, using suitable linear equations involving the entries of  $Q_j$  and  $\tilde{Q}_j$  and the  $a_i$ .

This is not yet quite a semidefinite feasibility problem as defined above, but it can be transformed into one by standard tricks. Namely, we first replace each of the scalar variables  $a_i$  by the difference  $a'_i - a''_i$ , where  $a'_i$  and  $a''_i$  are new nonnegative scalar variables. Then we set up a large block-diagonal matrix  $X$  that has the matrices  $Q_1, \dots, Q_m$  and  $\tilde{Q}_1, \dots, \tilde{Q}_m$  on the diagonal, as well as the  $1 \times 1$  blocks containing  $a'_1, a''_1, \dots, a'_m, a''_m$ , and zeros elsewhere. The zeros are forced as linear equalities, of the form  $A_j \bullet X = 0$ , for the entries of  $X$ . As is well known, positive semidefiniteness of  $X$  is equivalent to positive semidefiniteness of all the  $Q_j$  and  $\tilde{Q}_j$  plus the nonnegativity of the  $a'_i$  and  $a''_i$ . In this way, we get a semidefinite feasibility problem, whose input size is bounded by a polynomial in  $k$  and in the input size of the non-positivity problem.

We will refer to the resulting semidefinite feasibility problem as the *standard semidefinite formulation* of the non-positivity problem (or of the  $k$ -monotone interpolability problem).

**Feasible solutions requiring exponentially many digits.** Theorem 1.4, the example of a non-interpolable set for which a set lying extremely close is interpolable, yields the following consequence.

**Corollary 5.2.** *For the 3-monotone noninterpolable point set  $P_m$ ,  $m \geq 2$ , as in Theorem 1.4 (with  $O(m)$  points with integer coordinates bounded by  $O(m^3)$ ), every vector  $a$  as in the corresponding non-positivity problem (Problem 5.1) has entries exceeding  $2^{2^m}/100m$  in absolute value. Consequently, every feasible solution of the standard semidefinite formulation has components with exponentially many digits.*

*Proof.* Let  $a$  be a vector as in Problem 5.1, witnessing the 3-monotone non-interpolability of  $P_m$ , and let  $A = \|a\|_\infty = \max_i |a_i|$ .

Let  $P'_m$  be the 3-monotone interpolable set as in Theorem 1.4. Let  $v$  be the vector of the  $k$ th divided differences for  $P_m$  and  $v'$  the one for  $P'_m$ . Since the  $y$ -coordinates of  $P_m$  and of  $P'_m$  differ by at most  $\varepsilon := 2 \cdot 2^{-2^m}$  and the



$x$ -coordinates are integers, from the definition of divided differences it is easy to check that  $\|v - v'\|_\infty \leq 8\varepsilon$ .

Hence, with  $n = |P_m| = 2m + 3$ , we have  $\sum_{i=1}^n a_i v'_i \leq \sum_{i=1}^n a_i v_i + nA \cdot 8\varepsilon$ . If we had  $A \leq (8n\varepsilon)^{-1}$ , then  $\sum_{i=1}^n a_i v'_i < 0$ , and so  $a$  would also witness non-interpolability of  $P'_m$ . The corollary follows.  $\square$

**A simpler example for a variant of the non-positivity problem.** If we take the non-positivity problem for general quadratic polynomials  $\tilde{p}_{ij}$ , not necessarily coming from 3-monotone interpolability, there is a simpler example forcing exponentially many digits.

For simplicity, we replace the condition  $t \in [-1, 1]$  with  $t \in \mathbb{R}$ . A quadratic polynomial  $At^2 + Bt + C$  is nonnegative on  $\mathbb{R}$  if and only if  $A \geq 0$  and  $B^2 \leq 4AC$ .

Let us set  $v = (-1, 0, \dots, 0)$ ; then  $\sum_{i=1}^n a_i v_i$  forces  $a_1 = 1$ . Clearly, the polynomials  $\tilde{p}_{ij}(t)$  can be set so that the polynomials  $q_j(t) := \sum_{i=1}^n a_i \tilde{p}_{ij}(t)$  are as follows:  $q_1(t) = a_2 - 2a_1$ , and  $q_i(t) = a_{i+1}t^2 + 2a_i t + a_1$  for  $i = 2, 3, \dots, m$ . The nonnegativity of  $q_1$  makes sure that  $a_2 \geq 2$ , and nonnegativity of  $q_i$  yields  $4a_i^2 \leq 4a_{i+1}a_1$ . Then we have  $a_i \geq 2^{2^{i-2}}$ .

## Acknowledgment

We would like to thank Kirill Kopotun, Fadoua Balabdaoui, Jean B. Lasserre, and Mohb Safey El Din for kindly answering our questions, and Viola Meszáros for useful discussions at the initial stages of this research.

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